JOURNAL OF COMPUTATIONAL PHYSICS 10, 202–210 (1972)

On the Use of Nonuniform Grids in Finite-Difference Equations

EUGENIA KÁLNAY DE RIVAS*

Department of Meteorology, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Received August 2, 1971

A study of the truncation errors introduced by the use of nonuniform grids is presented. It is shown that the use of certain stretched coordinates has several advantages for the numerical study of flows with boundary layers.

Finite-difference schemes that use grids with uniform spacing are the simplest and most accurate, but they are not satisfactory in problems with boundary layers. If the number of points is not large enough to resolve the boundary layer (at least two or three points within it) then the numerical solution is apt to have gross errors even in the interior. The use of enough grid points to resolve the boundary layer then makes the total computation time unacceptably large. The problem can be solved by the introduction of an irregular net with smaller spacing near the boundary.

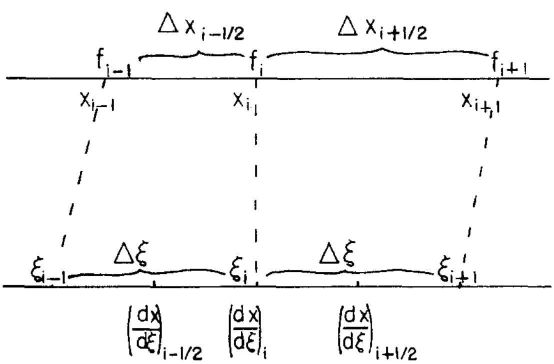

FIG. 1. Non-uniform grid defined through the use of a stretched coordinate.

* Present address: Laboratorio de Dinámica de la Atmósfera, Facultad de Ingeniería, Universidad de la República, Montevideo, Uruguay.

One possibility is to divide the grid intervals by two or more within the region of interest. This method has two disadvantages: first, it is necessary to interpolate values of the variables or their derivatives at intermediate points and weak numerical instabilities usually arise at the boundary between the large and small grid size, and second, this method cannot give really small grid intervals without greatly increasing the number of intermediate interpolations. Crowder and Dalton [4] have shown that, in a boundary-layer problem, the use of grids with discontinuously varying resolution gives worse overall errors than a regular grid with the same number of points. Another possibility is to vary the grid intervals continuously, avoiding the necessity of intermediate interpolations. Consider, for example, a function $f(x)$ defined on a nonuniform grid (Fig. 1).

Making a Taylor expansion about the center point x_i, there are two "centered" combinations of the functions at three points that give an appoximation of the first derivative f_i^{I}:

$$\begin{aligned}\frac{f_{i+1}-f_{i-1}}{\Delta x_{i+1/2}+\Delta x_{i-1/2}} &= f_i^{\mathrm{I}} + \frac{(\Delta x_{i+1/2}-\Delta x_{i-1/2})}{2} f_i^{\mathrm{II}} \\ &\quad + \frac{(\Delta x_{i+1/2})^2 - \Delta x_{i+1/2}\cdot\Delta x_{i-1/2} + (\Delta x_{i-1/2})^2}{6} f_i^{\mathrm{III}} + \cdots,\end{aligned} \tag{1}$$

which has first-order errors, and

$$\begin{aligned}&\frac{\Delta x_{i-1/2}}{(\Delta x_{i+1/2}+\Delta x_{i-1/2})\Delta x_{i+1/2}} f_{i+1} - \frac{\Delta x_{i+1/2}}{(\Delta x_{i+1/2}+\Delta x_{i-1/2})\Delta x_{i-1/2}} f_{i-1} \\ &\quad - \frac{\Delta x_{i+1/2}-\Delta x_{i-1/2}}{\Delta x_{i+1/2}\cdot\Delta x_{i-1/2}} f_i = f_i^{\mathrm{I}} + \frac{\Delta x_{i+1/2}\cdot\Delta x_{i-1/2}}{6} f_i^{\mathrm{III}} + \cdots,\end{aligned} \tag{2}$$

which has second-order errors. If $\Delta x_{i+1/2} = \Delta x_{i-1/2}$, both (1) and (2) reduce to the usual centered-difference scheme. However, there is only one combination of the three points that gives an approximation of the second derivative f_i^{II},

$$\begin{aligned}&\frac{f_{i+1}-f_i}{\Delta x_{i+1/2}[(\Delta x_{i+1/2}+\Delta x_{i-1/2})/2]} - \frac{f_i - f_{i-1}}{\Delta x_{i-1/2}[(\Delta x_{i+1/2}+\Delta x_{i-1/2})/2]} \\ &\quad = f_i^{\mathrm{II}} + \frac{\Delta x_{i+1/2}-\Delta x_{i-1/2}}{3} f_i^{\mathrm{III}} \\ &\qquad + \frac{(\Delta x_{i+1/2})^2 - \Delta x_{i+1/2}\cdot\Delta x_{i-1/2} + (\Delta x_{i-1/2})^2}{12} f_i^{\mathrm{IV}} + \cdots,\end{aligned} \tag{3}$$

which has first-order errors. Note that the second term of the right-hand side of Eq. (1) and (3) is the "extra error" introduced by the use of a nonuniform grid, while the following terms are equivalent to the second-order errors that are made when constant spacing is used.

Sundqvist and Veronis [8] reduced the "extra error" in Eq. (3) to second order by choosing the intervals such that

$$\Delta x_{i+1/2} - \Delta x_{i-1/2} = O(\Delta x_{i-1/2})^2 \tag{4}$$

but they still use (2) instead of (1). This method allows some improvement of the resolution near the boundary but it still requires a large number of points to reduce significantly the grid intervals there.

Suppose now that we vary the grid intervals by defining a stretched coordinate ξ,

$$x = x(\xi) \tag{5}$$

in such a way that the grid intervals $\Delta\xi$ are constant. If we are studying a function defined in a region $0 \leqslant x \leqslant 1$ with a boundary layer at $x = 0$, then $x(\xi)$ should have the following properties:

(a) $dx/d\xi$ should be finite over the whole interval. If $dx/d\xi$ becomes infinite at some point then the mapping $x = x(\xi)$ will give a poor resolution near that point, which cannot be improved by increasing the number of points, since

$$\Delta x \approx (dx/d\xi) \cdot \Delta\xi.$$

(b) $dx/d\xi = 0$ at $x = 0$. This will insure a high resolution near $x = 0$. Elsewhere $dx/d\xi$ should be different from zero.

Making a Taylor-series expansion of x about x_i we find

$$\begin{aligned}\Delta x_{i+1/2} &= x_{i+1} - x_i \\ &= \Delta\xi \left(\frac{dx}{d\xi}\right)_i + \frac{(\Delta\xi)^2}{2}\left(\frac{d^2x}{d\xi^2}\right)_i + \frac{(\Delta\xi)^3}{6}\left(\frac{d^3x}{d\xi^3}\right)_i + \frac{(\Delta\xi)^4}{24}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots \\ &= \Delta\xi \left(\frac{dx}{d\xi}\right)_{i+1/2} + \frac{(\Delta\xi)^3}{24}\left(\frac{d^3x}{d\xi^3}\right)_i + \frac{(\Delta\xi)^4}{48}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots,\end{aligned} \tag{6}$$

$$\begin{aligned}\Delta x_{i-1/2} &= x_i - x_{i-1} \\ &= \Delta\xi \left(\frac{dx}{d\xi}\right)_i - \frac{(\Delta\xi)^2}{2}\left(\frac{d^2x}{d\xi^2}\right)_i + \frac{(\Delta\xi)^3}{6}\left(\frac{d^3x}{d\xi^3}\right)_i - \frac{(\Delta\xi)^4}{24}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots \\ &= \Delta\xi \left(\frac{dx}{d\xi}\right)_{i-1/2} + \frac{(\Delta\xi)^3}{24}\left(\frac{d^3x}{d\xi^3}\right)_i - \frac{(\Delta\xi)^4}{48}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots,\end{aligned} \tag{7}$$

since

$$\left(\frac{dx}{d\xi}\right)_{i+1/2} = \left(\frac{dx}{d\xi}\right)_i + \frac{\Delta\xi}{2}\left(\frac{d^2x}{d\xi^2}\right)_i + \frac{(\Delta\xi)^2}{8}\left(\frac{d^3x}{d\xi^3}\right)_i + \frac{(\Delta\xi)^3}{48}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots,$$

$$\left(\frac{dx}{d\xi}\right)_{i-1/2} = \left(\frac{dx}{d\xi}\right)_i - \frac{\Delta\xi}{2}\left(\frac{d^2x}{d\xi^2}\right)_i + \frac{(\Delta\xi)^2}{8}\left(\frac{d^3x}{d\xi^3}\right)_i - \frac{(\Delta\xi)^3}{48}\left(\frac{d^4x}{d\xi^4}\right)_i + \cdots.$$

Therefore,

$$\Delta x_{i+1/2} + \Delta x_{i-1/2} = \Delta\xi\Big[2\Big(\frac{dx}{d\xi}\Big)_i + \frac{(\Delta\xi)^2}{3}\Big(\frac{d^3x}{d\xi^3}\Big)_i + \cdots\Big], \tag{8}$$

$$\Delta x_{i+1/2} - \Delta x_{i-1/2} = (\Delta\xi)^2\Big[\Big(\frac{d^2x}{d\xi^2}\Big)_i + \frac{(\Delta\xi)^2}{12}\Big(\frac{d^4x}{d\xi^4}\Big)_i + \cdots\Big]. \tag{9}$$

Introducing (6)–(9) into (1) and (3) we get

$$\begin{aligned}\frac{f_{i+1} - f_{i-1}}{2\Delta\xi(dx/d\xi)_i} &= f_i^{\mathrm{I}}\Big[1 + \frac{(\Delta\xi)^2}{6}\Big(\frac{d^3x}{d\xi^3}\Big)_i \Big/ \Big(\frac{dx}{d\xi}\Big)_i\Big] \\ &+ f_i^{\mathrm{II}}\frac{(\Delta\xi)^2}{2}\Big[\Big(\frac{d^2x}{d\xi^2}\Big)_i + \frac{(\Delta\xi)^2}{6}\Big\langle\frac{1}{2}\Big(\frac{d^4x}{d\xi^4}\Big)_i + \Big(\frac{d^3x}{d\xi^3}\Big)_i\cdot\Big(\frac{d^2x}{d\xi^2}\Big)_i\Big/\Big(\frac{dx}{d\xi}\Big)_i\Big\rangle\Big] \\ &+ f_i^{\mathrm{III}}\frac{(\Delta\xi)^2}{6}\Big[\Big(\frac{dx}{d\xi}\Big)_i^2 + \frac{(\Delta\xi)^2}{2}\Big\langle\frac{3}{2}\Big(\frac{d^2x}{d\xi^2}\Big)_i^2 + \Big(\frac{d^3x}{d\xi^3}\Big)_i\cdot\Big(\frac{dx}{d\xi}\Big)_i\Big\rangle\Big] + O(\Delta\xi)^3\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\frac{(f_{i+1} - f_i)\Big/\Big(\frac{dx}{d\xi}\Big)_{i+1/2} - (f_i - f_{i-1})\Big/\Big(\frac{dx}{d\xi}\Big)_{i-1/2}}{(\Delta\xi)^2\Big(\frac{dx}{d\xi}\Big)_i} \\ &= f_i^{\mathrm{II}}\Big[1 + \frac{5}{24}(\Delta\xi)^2\Big(\frac{d^3x}{d\xi^3}\Big)_i\Big/\Big(\frac{dx}{d\xi}\Big)_i\Big] \\ &+ f_i^{\mathrm{III}}\frac{(\Delta\xi)^2}{3}\Big[\Big(\frac{d^2x}{d\xi^2}\Big)_i + \frac{(\Delta\xi)^2}{12}\Big\langle\Big(\frac{d^4x}{d\xi^4}\Big)_i + \frac{5}{2}\Big(\frac{d^3x}{d\xi^3}\Big)_i\cdot\Big(\frac{d^2x}{d\xi^2}\Big)_i\Big/\Big(\frac{dx}{d\xi}\Big)_i\Big\rangle\Big] \\ &+ f_i^{\mathrm{IV}}\frac{(\Delta\xi)^2}{12}\Big[\Big(\frac{dx}{d\xi}\Big)_i + \frac{(\Delta\xi)^2}{4}\Big\langle 3\Big(\frac{d^2x}{d\xi^2}\Big)_i^2 + \frac{13}{6}\Big(\frac{d^3x}{d\xi^3}\Big)_i\cdot\Big(\frac{dx}{d\xi}\Big)_i\Big\rangle\Big] + O(\Delta\xi)^3.\end{aligned} \tag{11}$$

Equations (10) and (11) show that *any smooth function* $x(\xi)$ that satisfies conditions (a) and (b) will give an approximation of the first and second derivatives with second-order accuracy, since the "extra truncation errors" due to the nonuniformity of the grid *are of second order in* $\Delta\xi$. This useful result is true even near the boundary layer where the variations of Δx_i may be of the same order as Δx_i so that truncation errors may be of first order with respect to Δx_i [Eqs. (1) and (3)], but where the intervals Δx_i are very small so that the overall truncation error remains small.

The form of Eqs. (10) and (11) suggests the convenience of choosing a function $x = P_n(\xi)$, where P_n is a polynomial of degree greater than one, and in particular the advantage of the choice of the simple function (see Fig. 2)

$$x = \xi^2, \tag{12}$$

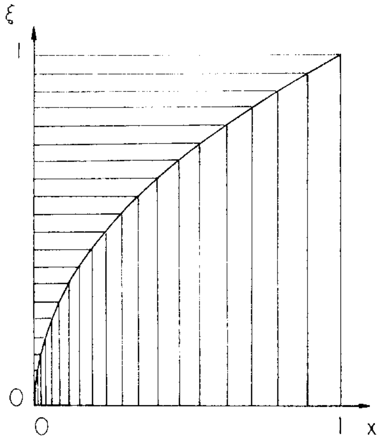

FIG. 2. Distribution of grid points when the stretched coordinate $x = \xi^2$ is used.

which has the following attractive properties:

(i) $$\max_{0 \leqslant x \leqslant 1} (dx/d\xi) = 2 \quad \text{at} \quad x = 1. \tag{13}$$

This implies that near $x = 1$, $\Delta x \approx 2/N$, where N is the number of intervals $N = 1/\Delta\xi$. This shows that at worst the stretched coordinate gives half the resolution of the uniform grid, which is not bad at all.

(ii) The first interior point at the boundary $x = 0$ will be such that

$$\Delta x_{1/2} = (\Delta\xi)^2 = 1/N^2. \tag{14}$$

Then the resolution near the boundary layer increases wtih N^2 and not with N as the uniform grid. The requirement of having, say, at least three grid points within a boundary layer of relative width d is fulfilled if $(3\Delta\xi)^2 < d$, or $N > 3/\sqrt{d}$ compared $N > 3/d$ in the case of a uniform grid.

(iii) $$d^2x/d\xi^2 = 2; \qquad d^3x/d\xi^3 = 0; \cdots. \tag{15}$$

Equations (10) and (11) reduce to

$$(f_{i+1} - f_i)\Big/\left[2(\Delta\xi)\left(\frac{dx}{d\xi}\right)_i\right] = f_i^{\mathrm{I}} + f_i^{\mathrm{II}}(\Delta\xi)^2 + f_i^{\mathrm{III}}\frac{2}{3}(\Delta\xi)^2\xi^2 + O(\Delta\xi)^3, \tag{16}$$

$$\frac{(f_{i+1} - f_i)\Big/\left(\frac{dx}{d\xi}\right)_{i+1/2} - (f_i - f_{i-1})\Big/\left(\frac{dx}{d\xi}\right)_{i-1/2}}{(\Delta\xi)^2\left(\frac{dx}{d\xi}\right)_i} \tag{17}$$

$$= f_i^{\mathrm{II}} + f_i^{\mathrm{III}}\frac{2}{3}(\Delta\xi)^2 + f_i^{\mathrm{IV}}\frac{(\Delta\xi)^2}{3}\xi^2 + O(\Delta\xi)^3 \tag{17}$$

and we see that the "extra truncation error" *is independent of* x (except for the variations of the derivatives of f).

When a boundary layer is expected both at $x = 0$ and at $x = 1$, a convenient stretched coordinate is defined by the symmetric function

$$x = \sin^2\left(\frac{\pi}{2}\xi\right), \tag{18}$$

with

$$dx/d\xi = \pi[x(1-x)]. \tag{19}$$

It behaves like $x = [(\pi/2)\xi]^2$ near $x = 0$, like $x = \{1 - [(\pi/2)(1-\xi)]^2\}$ near $x = 1$, and is rather linear in the interior (see Fig. 3). At the boundaries, the first interior point it at a distance

$$\Delta x \approx \pi^2/(4N^2) \tag{20}$$

and at the center point, $x = 0.5$,

$$\Delta x \approx \pi/(2N).$$

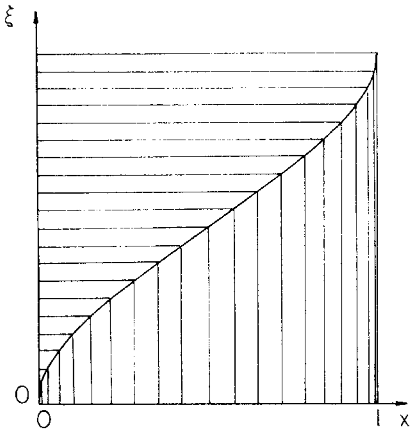

FIG. 3. Distribution of grid points when the stretched coordinate $x = \sin^2(\pi/2\ \xi)$ is used.

Both types of stretched coordinates have been successfully used in two-dimensional numerical models of the atmosphere of Venus (Kálnay de Rivas [5]).

We should point out here that in problems in which the equations of fluid motion are integrated with respect to time, the space intervals used in the criteria of linear computational instability should be, in general, the smallest Δx_i. The methods developed to avoid nonlinear computational instability can be easily adapted to nonuniform grids (see Arakawa [1] and Bryan [3]).

In some cases boundary conditions on the normal derivative of the function whose solution is being obtained, require the definition of a point immediately outside the boundary. This problem can be solved defining an external grid point located at the same distance from the boundary as the first interior point, even though this doesn't correspond to the definition of the stretched coordinate, which has $dx/d\xi = 0$ at the boundary. This poses no difficulty since at the boundary the normal derivative will be computed on a locally uniform grid with very small intervals, and therefore the truncation errors will be very small.

Finally we compare our results with those obtained by Sundqvist and Veronis [8] who solved numerically the following differential equation proposed by Stommel [7] for the wind-driven circulation in a homogeneous ocean:

$$\begin{aligned} \epsilon(\psi'' - \psi) + \psi' &= -\sin x \qquad \text{for} \quad \epsilon = 0.05, \\ \psi &= 0 \qquad \text{at} \quad x = 0, \pi, \end{aligned} \tag{21}$$

which has a boundary layer at $x = 0$. Sundqvist and Veronis set

$$\Delta x_{i+1/2} - \Delta x_{i-1/2} = (\alpha/\pi)(\Delta x_{i-1/2})^2, \qquad \alpha = 2. \tag{22}$$

The exact solution of (21) is also included in their paper.

Figure 4 compares the percentage errors introduced by the Sundqvist–Veronis grid with those introduced by using the stretched coordinate $x = \xi^2$. Note that not

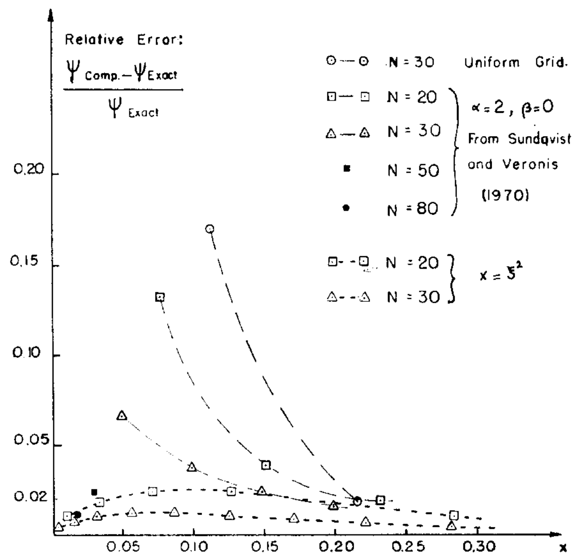

FIG. 4. Comparison of the results obtained by Sundqvist and Veronis and by using the stretched coordinate $x = \xi^2$.

only are the latter smaller, but that there is no tendency for the relative errors to grow as $x \to 0$, even though the function itself tends to zero.

If we compare (22) with (9) and (7) we see that the choice (22) corresponds approximately to the use of a stretched coordinate defined by the differential equation

$$d^2x/d\xi^2 = (\alpha/\pi)(dx/d\xi)^2, \tag{23}$$

which has the solution

$$\xi = e^{-(\alpha/\pi)x}. \tag{24}$$

Then the grid spacing is given by

$$\Delta x \approx (\pi/\alpha)[(1 - e^{-\alpha})/N] \qquad \text{near} \quad x = 0 \tag{25}$$

and

$$\Delta x \approx (\pi/\alpha)[(e^{\alpha} - 1)/N] \qquad \text{near} \quad x = \pi. \tag{26}$$

This is obviously not a good choice of $x(\xi)$ because as Sundqvist and Veronis pointed out, to obtain a good resolution near the origin, α should be large, and that would spoil the computations near $x = \pi$.

Another advantage of the method proposed here is that the actual spacing of the grid points is obtained immediately, once N is given, whereas the method proposed by Sundqvist and Veronis requires the solution of a rather cumbersome equation for $\Delta x_{1/2}$.

Beardsley [2] used the stretched coordinate $x = \xi^{1/2}$ to solve a problem with a boundary layer near $x = 1$. Since $dx/d\xi$, $d^2x/d\xi^2$,... $\to \infty$ as $x \to 0$, an inspection of Eq. (10) and (11) shows clearly why the truncation errors that he obtained were very large near $x = 0$. Near $x = 1$ this stretched coordinate gives $\Delta x \approx 1/(2N)$, so that it only increases by 2 the resolution of a regular grid.

Roberts [6] has developed a more general though less simple method similar to the one presented here. He defines a stretched coordinate by superimposing a suitably chosen function of the family of logarithms which gives the high resolution needed in the boundary on a linear function of the physical coordinate which gives enough resolution in the interior.

Acknowledgments

I would like to thank Professor J. G. Charney for suggesting the use of stretched coordinates in the study of the circulation of the atmosphere of Venus. This research was supported by the National Science Foundation (Grant GA-402X).

REFERENCES

1. A. ARAKAWA, Computational design for long-term numerical integration of the equations of fluid motion: two-dimensional incompressible flow: Part I, *J. Comput. Phys.* **1** (1966), 119–143.
2. R. C. BEARDSLEY, Integration of the planetary vorticity equation on a parabolic circular grid, *J. Comput. Phys.* **7** (1971), 273–288.
3. K. BRYAN, A scheme for numerical integration of the equations of motion on an irregular grid free from nonlinear instability, *Mon. Weather Rev.* **9** (1966), 39–40.
4. H. J. CROWDER AND C. DALTON, Errors in the use of non-uniform mesh systems, *J. Comput. Phys.* **7** (1971), 32–45.
5. E. KÁLNAY DE RIVAS, The circulation of the atmosphere of Venus, Ph.D. Thesis, Department of Meteorology, Mass. Inst. of Tech, MA, 1971.
6. G. I. ROBERTS, Computational meshes for boundary layer problems, *in* "Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics," (M. Holt, Ed.), Springer-Verlag, Berlin-Heidelberg-New York, 1971.
7. H. STOMMEL, The westward intensification of wind-driven ocean currents, *Trans. Am. Geophys. Union* **29** (1948), 202–206.
8. H. SUNDQVIST AND G. VERONIS, A simple finite-difference grid with nonconstant intervals, *Tellus* **22** (1970), 26–31.